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An Extremal Relation of the Theory of Approximation of Functions by Algebraic Polynomials

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In this paper we find an extremal relation of the theory of approximation of functions on a finite interval by algebraic polynomials. In the Legendre case we find some estimations and representation in the form of a series of the constant from the extremal relation.

1. INTRODUCTION

In the theory of approximation of functions by polynomials there is a series of publications which have considered approximation of periodic functions, represented in the form of convolution, by trigonometric polynomials of order not exceeding n [1, 7–9, 18, 29, 31]. In these publications, in particular, important extremal relations of the theory of approximation of periodic functions by trigonometric polynomials can be found (for example, the least upper bounds of the best approximations by trigonometric polynomials of order not exceeding n in the metrics of C [1, 8, 9] and L [18] of certain classes of differentiable functions and the best linear methods for these classes).

In this paper we have made steps towards constructing an analogous theory in the case of approximation of functions, defined on a finite interval, by algebraic polynomials. Making use of the technique of Jacobi polynomials, we obtain an extremal relation of the theory of approximation of functions by algebraic polynomials.

We consider the generalized translation operator defined on the space of functions summable on $[-1, 1]$ with the weight $(1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha \geqslant \beta \geqslant -\frac{1}{2})$ and introduce a generalized convolution corresponding to this operator. We also introduce some differential operators which in the algebraic case are some analogues of derivatives; it is worth noticing that the Jacobi polynomials are eigenfunctions of these differential operators. Furthermore, we consider some classes of functions, representable in the form of generalized convolution.

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Making use of some relations for these classes which are similar to the duality relations established in [18] for the classes of usual convolutions, defined on the space of all summable 2π-periodic functions, we obtain for these classes an extremal relation of the theory of approximation of functions by algebraic polynomials. It seems worthwhile that this relation holds for the functional class which is substantially more extensive than the well-known class W_L^{2r} (see the definition of this class in Section 9; we are now talking about the case $\alpha=\beta=0$).

We note that some statements from this paper were published without proof in [27].

2. GENERALIZED TRANSLATION OPERATOR. GENERALIZED CONVOLUTION

Let $\{J_n^{(\alpha,\beta)}\}_0^\infty=\{J_n\}_0^\infty$ be the orthonormal system of Jacobi polynomials on $[-1,1]$ with the weight $p(x)=(1-x)^\alpha(1+x)^\beta$ $(\alpha,\beta>-1)$. We denote by $L_{p;\alpha,\beta}$ $(1\leqslant p<\infty)$ the space of functions f, Lebesgue-measurable on $[-1,1]$, such that

$$\|f\|_{p;\alpha,\beta}=\left\{\int_{-1}^{1} p(x)\,|f(x)|^p\,dx\right\}^{1/p}<\infty;$$

$L_{1;\alpha,\beta}=L_{\alpha,\beta}$, $L_{0,0}=L$, $L_{\infty;\alpha,\beta}=C[-1,1]=C$. For $f\in L_{\alpha,\beta}$ we denote by $\{c_k^{\alpha,\beta}(f)\}_0^\infty=\{c_k(f)\}_0^\infty$ its Fourier coefficients with respect to the system $\{J_k\}_0^\infty$.

If $\alpha\geqslant\beta\geqslant-\frac{1}{2}$, then there exists $\forall t\in[0,\pi]$ an operator $f\to f_t$, defined on $L_{\alpha,\beta}$, with the following properties:

(1) $\forall f\in L_{p;\alpha,\beta}$ $(p\in[1,\infty])$ and $\forall t\in[0,\pi]$ we have $f_t\in L_{p;\alpha,\beta}$; moreover

$$\|f_t\|_{p;\alpha,\beta}\leqslant\|f\|_{f;\alpha,\beta}; \tag{1}$$

(2) $\forall f\in L_{\alpha,\beta}$ and $\forall t\in[0,\pi]$ we have

$$c_k(f_t)=c_k(f)\cdot J_k(\cos t)(J_k(1))^{-1},\qquad k+1\in N. \tag{2}$$

The existence of the operator $f\to f_t$ possessing these properties has been proved in [10]. The integral representation of f_t in the ultraspherical case $\alpha=\beta$ can be traced back to [12]. In the general Jacobi case, an integral representation was first given by G. Gasper [10, 11] and in a different form by T. Koornwinder [16]. The operator $f \to f_t$ will be called a generalized translation operator (g.t.o.). It follows from the integral representations for f_t mentioned above that if f is finite on $[-1, 1]$, then $\forall x \in [-1, 1]$ and $\forall t \in [0, \pi]$ we have $f_t(x) = f_{\arccos x}(\cos t)$. The g.t.o. has been considered in [3, 4, 6, 19–23, 26, 28, 30, 34] in connection with some problems of approximation of functions on a finite interval by algebraic polynomials.

We introduce for $\varphi, g \in L_{\alpha, \beta}$ the function

$$(\varphi * g)(x) = \int_0^\pi \varphi_t(x)\, g(\cos t)\, p(\cos t) \sin t\, dt; \tag{3}$$

it is easy to verify that $\varphi * g \in L_{\alpha, \beta}$. We will call $\varphi * g$ the generalized convolution (g.c.) of the functions φ and g.

We list some properties of the g.c.:

(1) $\forall \varphi, g \in L_{\alpha, \beta}$ we have

$$c_k(\varphi * g) = c_k(\varphi)\, c_k(g)(J_k(1))^{-1}, \qquad k + 1 \in N; \tag{4}$$

(2) if $1 \leqslant p,\ q < \infty,\ p^{-1} + q^{-1} > 1,\ r^{-1} = p^{-1} + q^{-1} - 1,\ \varphi \in L_{p; \alpha, \beta}$, $g \in L_{q; \alpha, \beta}$, then $\varphi * g \in L_{r; \alpha, \beta}$ and

$$\|\varphi * g\|_{r; \alpha, \beta} \leqslant \|\varphi\|_{p; \alpha, \beta} \cdot \|g\|_{q; \alpha, \beta}; \tag{5}$$

(3) if $1 \leqslant p,\ q \leqslant \infty,\ \varphi \in L_{p; \alpha, \beta},\ g \in L_{q; \alpha, \beta},\ p^{-1} + q^{-1} = 1$, then $\varphi * g \in C$ and

$$\sup\{\|\varphi * g\|_C : \varphi \in L_{p; \alpha, \beta}, \|\varphi\|_{p; \alpha, \beta} \leqslant 1\} = \|g\|_{q; \alpha, \beta} \tag{6}$$

(here and below we assume that $0^{-1} = \infty$, $\infty^{-1} = 0$).

Equality (4) follows directly from (2) and (3). Facts analogous to (2) and (3) in case of the ordinary convolution, defined on the class of all L-summable 2π-periodic functions, are well known [35, p. 71], [17, pp. 71–72]; properties (2) and (3) of the g.c. can be proved in a similar manner. In case $\alpha = \beta = 0$ they have been proved in [26]. We note that, as it follows directly from (4), the operation of taking the g.c. is commutative and associative.

3. SOME DUALITY RELATIONS FOR CLASSES OF THE G.C.

We formulate the following theorem due to S. M. Nikol'skii [18].

THEOREM. *Let F be a finite-dimensional subspace of $L_{p;\alpha,\beta}$ ($p\in[1,\infty]$), $f\in L_{p;\alpha,\beta}$. The following relation holds*

$$\min\{\|f-u\|_{p;\alpha,\beta}\colon u\in F\}=\sup\left\{\int_{-1}^{1} p(x)\,f(x)\,v(x)\,dx\colon v\in G_{q;\alpha,\beta}(F)\right\}, \tag{7}$$

where $p^{-1}+q^{-1}=1$, $G_{q;\alpha,\beta}(F)=\{v\colon \|v\|_{q;\alpha,\beta}\leqslant 1,\ v\perp F\}$ (notation $v\perp F$ means that $\forall h\in F$ we have $\int_{-1}^{1} p(x)\,h(x)\,v(x)\,dx=0$).

LEMMA 1. *Let us assume that (1) $f=\varphi * g$, where g is fixed, (2) either $1\leqslant q\leqslant p\leqslant\infty$, $g\in L_{\alpha,\beta}$ or $q\geqslant p$, $g\in L_{s;\alpha,\beta}$ ($s^{-1}=q^{-1}-p^{-1}+1$), (3) F is a finite-dimensional subspace of C, (4) $p^{-1}+p'^{-1}=q^{-1}+q'^{-1}=1$. Let $E(f,F)_{q;\alpha,\beta}$ be the best approximation of f by elements from F in the $L_{q;\alpha,\beta}$-metric. Then we have*

$$\sup\{E(f,F)_{q;\alpha,\beta}\colon \|\varphi\|_{p;\alpha,\beta}\leqslant 1\}=\sup\{\|f\|_{p';\alpha,\beta}\colon \varphi\in G_{q';\alpha,\beta}(F)\}. \tag{8}$$

Proof. If $\varphi\in L_{p;\alpha,\beta}$, $1\leqslant q\leqslant p\leqslant\infty$, $g\in L_{\alpha,\beta}$, then, due to property (2) of the g.c., we get $f\in L_{p;\alpha,\beta}\subset L_{q;\alpha,\beta}$; if $q\geqslant p$, $g\in L_{s;\alpha,\beta}$ ($s^{-1}=q^{-1}-p^{-1}+1$), then, by making use of the same property (2) of the g.c., we obtain $f\in L_{r;\alpha,\beta}$, where $r^{-1}=p^{-1}+s^{-1}-1=p^{-1}+q^{-1}-p^{-1}+1-1=q^{-1}$, so that $r=q$ and $f\in L_{q;\alpha,\beta}$.

Obviously, we can assume g to be finite on $[-1,1]$. Taking into consideration (7), the commutative property of the g.c., and the expression for the norm of a linear functional on $L_{p;\alpha,\beta}$, we obtain

$$\begin{aligned}
&\sup\{E(f,F)_{q;\alpha,\beta}\colon \|\varphi\|_{p;\alpha,\beta}\leqslant 1\}\\
&\quad=\sup\left\{\sup\left\{\int_{-1}^{1} p(x)\,h(x)\,f(x)\,dx\colon h\in G_{q';\alpha,\beta}(F)\right\}\colon \|\varphi\|_{p;\alpha,\beta}\leqslant 1\right\}\\
&\quad=\sup\left\{\sup\left\{\int_{-1}^{1} p(x)\,h(x)\,dx\int_0^{\pi}\varphi_t(x)\,g(\cos t)\,p(\cos t)\sin t\,dt:\right.\right.\\
&\qquad\qquad\left.\left. h\in G_{q';\alpha,\beta}(F)\right\}:\|\varphi\|_{p;\alpha,\beta}\leqslant 1\right\}\\
&\quad=\sup\left\{\sup\left\{\int_{-1}^{1} p(x)\,h(x)\,dx\int_0^{\pi} g_t(x)\,\varphi(\cos t)\,p(\cos t)\sin t\,dt:\right.\right.\\
&\qquad\qquad\left.\left. h\in G_{q';\alpha,\beta}(F)\right\}:\|\varphi\|_{p;\alpha,\beta}\leqslant 1\right\}
\end{aligned}$$

$$= \sup \left\{ \sup \left\{ \int_{-1}^{1} p(x)\, h(x)\, dx \int_{-1}^{1} g_{\arccos z}(x)\, \varphi(z)\, p(z)\, dz : \right.\right.$$

$$\left.\left. h \in G_{q';\,\alpha,\,\beta}(F) \right\} : \|\varphi\|_{p;\,\alpha,\,\beta} \leqslant 1 \right\}$$

$$= \sup \left\{ \sup \left\{ \int_{-1}^{1} \varphi(z)\, p(z)\, dz \int_{-1}^{1} g_{\arccos z}(x)\, h(x)\, p(x)\, dx : \right.\right.$$

$$\left.\left. \|\varphi\|_{p;\,\alpha,\,\beta} \leqslant 1 \right\} : h \in G_{q';\,\alpha,\,\beta}(F) \right\}$$

$$= \sup\{ \|g * h\|_{p';\,\alpha,\,\beta} \colon h \in G_{q';\,\alpha,\,\beta}(F)\}$$

$$= \sup\{ \|f\|_{p';\,\alpha,\,\beta} \colon \varphi \in G_{q';\,\alpha,\,\beta}(F)\}.$$

The lemma is proved.

Let H_n be the set of all algebraic polynomials of degree at most n. For $f \in L_{q;\,\alpha,\,\beta}$ ($q \in [1, \infty]$) we introduce

$$E_n(f)_{q;\,\alpha,\,\beta} = \min\{ \|f - Q_n\|_{q;\,\alpha,\,\beta} \colon Q_n \in H_n\}.$$

It follows from Lemma 1 that

(1) the following relation holds:

$$\sup\{ E_n(f)_{q;\,\alpha,\,\beta} \colon \|\varphi\|_{p;\,\alpha,\,\beta} \leqslant 1\} = \sup\{ \|f\|_{p';\,\alpha,\,\beta} \colon \varphi \in G_{q';\,\alpha,\,\beta}(H_n)\}; \tag{9}$$

(2) if $q \in [1, \infty]$, $g \in L_{q;\,\alpha,\,\beta}$, $f = \varphi * g$, $q^{-1} + q'^{-1} = 1$, then

$$\sup\{ E_n(f)_{q;\,\alpha,\,\beta} \colon \|\varphi\|_{1;\,\alpha,\,\beta} \leqslant 1\} = \sup\{ \|f\|_{C} \colon \varphi \in G_{q';\,\alpha,\,\beta}(H_n)\}; \tag{10}$$

(3) if $q \in [1, \infty]$, $g \in L_{q;\,\alpha,\,\beta}$, $f = \varphi * g$, then

$$\sup\{ \|f\|_{q;\,\alpha,\,\beta} \colon \|\varphi\|_{1;\,\alpha,\,\beta} \leqslant 1\} = \|g\|_{q;\,\alpha,\,\beta}. \tag{11}$$

We need to prove only part (3). If in Lemma 1 we take $p = 1$ and $\mathscr{F} = \{0\}$, then we obtain $E(f, \mathscr{F})_{q;\,\alpha,\,\beta} = \|f\|_{q;\,\alpha,\,\beta}$; $G_{q';\,\alpha,\,\beta}(\mathscr{F})$ is the unit ball in $L_{q';\,\alpha,\,\beta}$ with center O. According to Lemma 1,

$$\sup\{ \|f\|_{q;\,\alpha,\,\beta} : \|\varphi\|_{1;\,\alpha,\,\beta} \leqslant 1\} = \sup\{ \|f\|_{C} : \|\varphi\|_{q';\,\alpha,\,\beta} \leqslant 1\}.$$

It remains to make use of (6).

LEMMA 2. *If* $f=\varphi * g$, $g\in L_{q;\alpha,\beta}$ $(1\leqslant q\leqslant\infty)$, $n+1\in N$, $q^{-1}+q'^{-1}=1$, *then*

$$\sup\{\|f\|_C\colon \varphi\in G_{q';\alpha,\beta}(H_n)\}=E_n(g)_{q;\alpha,\beta}. \tag{12}$$

A statement analogous to Lemma 2 for ordinary convolutions is well known [18]; see also [17, p. 78]. Lemma 2 can be proved in a similar manner.

COROLLARY 1. *If* $f=\varphi * g$, $g\in L_{q;\alpha,\beta}$ $(1\leqslant q\leqslant\infty)$, $n+1\in N$, *then*

$$\sup\{E_n(f)_{q;\alpha,\beta}\colon \|\varphi\|_{1;\alpha,\beta}\leqslant 1\}=E_n(g)_{q;\alpha,\beta}. \tag{13}$$

For the proof we compare (10) *and* (12).

LEMMA 3. *Let* $f=\varphi * g$, $1\leqslant p$, $q\leqslant\infty$, $p^{-1}+q^{-1}\geqslant 1$, $r^{-1}=p^{-1}+q^{-1}-1$, $\varphi\in L_{p;\alpha,\beta}$, $g\in L_{q;\alpha,\beta}$, $n+1\in N$. *Then we have*

$$E_n(f)_{r;\alpha,\beta}\leqslant E_n(\varphi)_{p;\alpha,\beta}\cdot E_n(g)_{q;\alpha,\beta}. \tag{14}$$

This statement is analogous to the well-known inequality of Sun' Yun-Shen [31] (*see also* [33, *p.* 316]) *and can be proved in a similar fashion.*

4. THE FUNCTIONS $\Phi_{r;\alpha,\beta}$

Let $r\in N$, $t\in(-1,1)$. We introduce the function

$$\Phi_{r;\alpha,\beta}(t)=\Phi_r(t)=(-1)^r\,2^{-\alpha-\beta-r}\Gamma(r+\alpha+\beta+1)\,\Gamma^{-2}(r)\,\Gamma^{-1}(\alpha+1)$$
$$\times\Gamma^{-1}(\beta+r)\int_{-1}^{t}(1-z)^{-\alpha-1}\,(1+z)^{-\beta-r}\,(t-z)^{r-1}\,dz$$
$$\times\int_{-1}^{z}(1-u)^{\alpha}\,(1+u)^{\beta+r-1}\,du.$$

Later in this paper we will prove a theorem on the representation of any function f from some class essentially in the form of the g.c. of some differential operator $\mathscr{D}_{r;\alpha,\beta}(f)$ and the function $\Phi_{r;\alpha,\beta}$. Thus, the function $\Phi_{r;\alpha,\beta}$ will play the role of a kernel in this representation. It will be shown below that $c_k(\mathscr{D}_{r;\alpha,\beta}(f))$ $(k+1\in N)$ are certain multiples of $c_k(f)$ $(k+1\in N)$, respectively, and the function $\Phi_{r;\alpha,\beta}$ is constructed so that the g.c. of $\mathscr{D}_{r;\alpha,\beta}(f)$ and $\Phi_{r;\alpha,\beta}$ coincides essentially with f.

LEMMA 4. *If* $r>\alpha+1$, *then* $\Phi_r\in C$; *if* $r=\alpha+1$, *then* $\Phi_r\in L_{q;\alpha,\beta}$ $(1\leqslant q<\infty)$; *if* $r<\alpha+1$, *then* $\Phi_r\in L_{q;\alpha,\beta}$ $(1\leqslant q<\frac{1+\alpha}{1+\alpha-r})$.

We omit the simple proof of this lemma.

DEFINITION. We say that r, α and q are consistent if either $r>\alpha+1$, $q\in[1,\infty]$, or $r=\alpha+1$, $q\in[1,\infty)$, or $r<\alpha+1$, $q\in[1,\frac{1+\alpha}{1+\alpha-r})$.

LEMMA 5. *If* $k, r\in N$, $k\geqslant r$, *then*

$$c_k(\Phi_r)=(-1)^r\,C_1(k,r,\alpha,\beta)\,J_k(1), \tag{15}$$

where $C_1(k,r,\alpha,\beta)=\Gamma(k-r+1)\,\Gamma(k+\alpha+\beta+1)\,\Gamma^{-1}(k+1)\,\Gamma^{-1}(k+\alpha+\beta+r+1)$ (*here and below, by* C_m *with parameters listed inside the parentheses we mean positive constants depending on these parameters*).

Proof. We introduce the following notation:

$$\lambda(z)=\int_{-1}^{z}(1-u)^{\alpha}\,(1+u)^{\beta+r-1}\,du,$$

$$\begin{aligned}C_2(\alpha,\beta,r)=C_2=(-1)^r\,\Gamma(\alpha+\beta+r+1)\,2^{-\alpha-\beta-r}\Gamma^{-2}(r)\\ \times\Gamma^{-1}(\alpha+1)\,\Gamma^{-1}(\beta+r),\end{aligned} \tag{16}$$

$$K_{r;\alpha,\beta}(t)=K_r(t)=C_2\cdot\Gamma^2(r)(1-t)^{r-1}\,\lambda(t).$$

Since

$$\Phi_r(t)=C_2\cdot\int_{-1}^{t}(1-z)^{-\alpha-1}\,(1+z)^{-\beta-r}\,\lambda(z)(t-z)^{r-1}\,dz,$$

we conclude that for $t\in(-1,1)$ we have

$$\begin{aligned}\Phi_r^{(r)}(t)&=C_2\cdot\Gamma(r)(1-t)^{-\alpha-1}\,(1+t)^{-\beta-r}\,\lambda(t)\\ &=\Gamma^{-1}(r)(1-t)^{-\alpha-r}\,(1+t)^{-\beta-r}\,K_r(t).\end{aligned} \tag{17}$$

As it follows directly from (16),

$$\begin{aligned}&K_r^{(m)}(1)=0\quad(m=\overline{0,r-2}),\qquad K_r^{(r-1)}(1)=-\Gamma(r)\quad(r\geqslant 2);\\ &K_1(1)=-1.\end{aligned} \tag{18}$$

We will prove that

$$K_r^{(m)}(-1)=0 \qquad (m=\overline{0, r-1}). \tag{19}$$

For $m=0$ relation (19) is obvious and so we can assume that $1 \leqslant m \leqslant r-1$. We have

$$\begin{aligned}
K_r^{(m)}(t) &= C_2 \Gamma^2(r) \sum_{i=0}^{m} \binom{m}{i} \lambda^{(i)}(t)((1-t)^{r-1})^{(m-i)} \\
&= C_2 \Gamma^2(r) \Big\{ \lambda(t)((1-t)^{r-1})^{(m)} \\
&\quad + \sum_{i=1}^{m} \binom{m}{i} ((1-t)^{\alpha} (1+t)^{\beta+r-1})^{(i-1)} ((1-t)^{r-1})^{(m-i)} \Big\} \\
&= C_2 \Gamma^2(r) \Big[\lambda(t)((1-t)^{r-1})^{(m)} + \sum_{i=1}^{m} \binom{m}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} \\
&\quad \times ((1+t)^{\beta+r-1})^{(j)} ((1-t)^{\alpha})^{(i-1-j)} ((1-t)^{r-1})^{(m-i)} \Big] \\
&= C_2 \Gamma^2(r) \Big[\lambda(t)((1-t)^{r-1})^{(m)} + \sum_{i=1}^{m} \binom{m}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} \prod_{k=1}^{j} (\beta+r-k) \\
&\quad \cdot (1+t)^{\beta+r-1-j} ((1-t)^{\alpha})^{(i-1-j)} ((1-t)^{r-1})^{(m-i)} \Big],
\end{aligned}$$

which implies (19). Making use of (17), integrating r times by parts, taking into account (18), (19) and the formula

$$(J_k^{(\alpha, \beta)})^{(r)} = C_1^{-1/2}(k, r, \alpha, \beta)\, J_{k-r}^{(\alpha+r, \beta+r)}, \qquad k \geqslant r, \tag{20}$$

[32, formulas (4.21.7) and (4.3.4)], we obtain

$$\begin{aligned}
c_{k-r}^{(\alpha+r, \beta+r)}(\Phi_r^{(r)}) &= C_1^{1/2}(k, r, \alpha, \beta)\, \Gamma^{-1}(r) \Big\{ (-1)^r\, \Gamma(r)\, J_k(1) \\
&\quad + (-1)^r\, \Gamma^2(r)\, C_2 \cdot \int_{-1}^{1} J_k(t) \cdot [(1-t)^{r-1} \lambda(t)]^{(r)}\, dt \Big\}.
\end{aligned} \tag{21}$$

Furthermore,

$$
\begin{aligned}
[(1-t)^{r-1}\lambda(t)]^{(r)} &= \sum_{i=0}^{r} \binom{r}{i} \lambda^{(i)}(t)((1-t)^{r-1})^{(r-i)} \\
&= \sum_{i=0}^{r} \binom{r}{i} ((1-t)^{\alpha}(1+t)^{\beta+r-1})^{(i-1)} \cdot (-1)^{r-i} \\
&\quad \times \prod_{k=0}^{r-i-1} (r-1-k)(1-t)^{i-1} \\
&= \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} \prod_{k=0}^{r-i-1} (r-1-k)(1-t)^{i-1} \\
&\quad \times \sum_{j=0}^{i-1} \binom{i-1}{j} ((1-t)^{\alpha})^{(j)} ((1+t)^{\beta+r-1})^{(i-1-j)} \\
&= \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} \prod_{k=0}^{r-i-1} (r-1-k)(1-t)^{i-1} \\
&\quad \times \sum_{j=0}^{i-1} \binom{i-1}{j} \prod_{l=0}^{j-1} (\alpha-l)(-1)^{j} (1-t)^{\alpha-j} \\
&\quad \times \prod_{p=0}^{i-j-2} (\beta+r-1-p)(1+t)^{\beta+r-i+j} \\
&= (1-t)^{\alpha}(1+t)^{\beta} Q_{r-1}(t), \qquad (22)
\end{aligned}
$$

where $Q_{r-1} \in H_{r-1}$ (if the upper index in a product $\prod$ is smaller than the lower one, then the product is considered to be equal to 1).

From (21) and (22) it follows that

$$c_{k-r}^{(\alpha+r,\beta+r)}(\Phi_r^{(r)}) = C_1^{1/2}(k, r, \alpha, \beta) \cdot (-1)^r J_k(1), \qquad k \geqslant r. \qquad (23)$$

Integrating r times by parts and taking into account the formula

$$
\begin{aligned}
&((1-x)^{\alpha+r}(1+x)^{\beta+r} J_{k-r}^{(\alpha+r,\beta+r)}(x))^{(r)} \\
&\quad = (-1)^r C_1^{-1/2}(k, r, \alpha, \beta)(1-x)^{\alpha}(1+x)^{\beta} J_k^{(\alpha,\beta)}(x), \qquad k \geqslant r \qquad (24)
\end{aligned}
$$

[32, formulas (4.10.1) and (4.3.4)], we find that

$$c_{k-r}^{(\alpha+r,\beta+r)}(\Phi_r^{(r)}) = C_1^{-1/2}(k, r, \alpha, \beta)\, c_k^{(\alpha,\beta)}(\Phi_r), \qquad k \geqslant r. \qquad (25)$$

Comparing (23) and (25), we obtain (15). The lemma is proved.

5. REPRESENTATION OF FUNCTIONS IN THE FORM OF THE G.C.

We denote by $AC[a,b]$ the class of absolutely continuous functions on $[a,b]$. For $r\in N$ we define

$$\Omega_{r;\alpha,\beta}=\Omega_r=[f\colon \exists f^{(2r-1)}(x)\ \forall x\in(-1,1),$$
$$\psi_{f,r}^{[k]}(x)=\begin{cases}((1-x)^{\alpha+r}(1+x)^{\beta+r}f^{(r)}(x))^{(k)}, & |x|<1\\ 0, & |x|=1\end{cases}$$
$$\in AC[-1,1],\quad k=\overline{0,r-1}].$$

For $f\in\Omega_r$ we introduce

$$\mathscr{D}_{r;\alpha,\beta}(f;x)=\mathscr{D}_r(f;x)$$
$$=(1-x)^{-\alpha}(1+x)^{-\beta}((1-x)^{\alpha+r}(1+x)^{\beta+r}f^{(r)}(x))^{(r)}.$$

It is clear that $\mathscr{D}_r(f)\in L_{\alpha,\beta}$.

LEMMA 6. *If $f\in\Omega_r$, then for $i=\overline{1,r}$ the following relation holds*

$$\lim_{x\to\pm1}[f^{(r-i)}(x)(1-x)^{\alpha+r-i+1}(1+x)^{\beta+r-i+1}]=0. \tag{26}$$

We omit the proof. It is based on Taylor's formula for $f^{(r-i)}$ with the remainder in the form of a definite integral.

LEMMA 7. *For $f\in\Omega_r$ we have*

$$c_k(D_r(f))=\begin{cases}(-1)^r C_1^{-1}(k,r,\alpha,\beta)\,c_k(f), & k\geqslant r,\\ 0 & 0\leqslant k\leqslant r-1.\end{cases} \tag{27}$$

Proof. Taking into account the definition of the class Ω_r and integrating r times by parts, we obtain

$$c_k(\mathscr{D}_r(f))=(-1)^r\int_{-1}^{1}(1-x)^{\alpha+r}(1+x)^{\beta+r}f^{(r)}(x)\,J_k^{(r)}(x)\,dx, \tag{28}$$

which implies at once the second line in (27). Assume now that $k\geqslant r$. We notice now that $\Omega_r\subset L_{\alpha,\beta}$; it can be proved by using Taylor's formula for f with the remainder written in the form of a definite integral. Making use of formulas (20) and (24) and denoting $v(x)=(1-x)^{\alpha+r}(1+x)^{\beta+r}\times J_{k-r}^{(\alpha+r,\beta+r)}(x)$, from (28) we obtain

$$
\begin{aligned}
c_k(\mathscr{D}_r(f)) &= (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \\
&\quad \times \int_{-1}^{1} (1-x)^{\alpha+r}\,(1+x)^{\beta+r}\, f^{(r)}(x)\, J_{k-r}^{(\alpha+r,\,\beta+r)}(x)\, dx \\
&= (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \int_{-1}^{1} v(x)\, f^{(r)}(x)\, dx \\
&= (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \lim_{\delta \to 0^+} \int_{-1+\delta}^{1-\delta} f^{(r)}(x)\, v(x)\, dx \\
&= (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \\
&\quad \times \lim_{\delta \to 0^+} \left\{ \sum_{i=1}^{r} (-1)^{i-1} f^{(r-i)}(x)\, v^{(i-1)}(x) \Big|_{-1+\delta}^{1-\delta} \right. \\
&\quad \left. + (-1)^r \int_{-1+\delta}^{1-\delta} f(x)\, v^{(r)}(x)\, dx \right\} \\
&= (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \sum_{i=1}^{r} (-1)^{i-1} \\
&\quad \times \lim_{\delta \to 0^+} [f^{(r-i)}(x)\, v^{(i-1)}(x)] \Big|_{-1+\delta}^{1-\delta} + (-1)^r\, C_1^{-1}(k, r, \alpha, \beta) \\
&\quad \times \lim_{\delta \to 0^+} \int_{-1+\delta}^{1-\delta} (1-x)^{\alpha}\,(1+x)^{\beta}\, J_k(x)\, f(x)\, dx \\
&= (-1)^r\, C_1^{-1}(k, r, \alpha, \beta)\, c_k(f) + (-1)^r\, C_1^{-1/2}(k, r, \alpha, \beta) \\
&\quad \times \sum_{i=1}^{r} (-1)^{i-1} \lim_{\delta \to 0^+} [f^{(r-i)}(x)\, v^{(i-1)}(x)] \Big|_{-1+\delta}^{1-\delta}. \qquad (29)
\end{aligned}
$$

Further, it is obvious that

$$
v^{(i-1)}(x) = (1-x)^{\alpha+r-i+1}\,(1+x)^{\beta+r-i+1}\, Q_i(x), \qquad 1 \leqslant i \leqslant r, \qquad (30)
$$

Q_i $(1 \leqslant i \leqslant r)$ being polynomials. Taking into account (26) and (30), we derive

$$
\lim_{\delta \to 0^+} [f^{(r-i)}(x)\, v^{(i-1)}(x)] \Big|_{-1+\delta}^{1-\delta} = 0, \qquad 1 \leqslant i \leqslant r. \qquad (31)
$$

The first line in (27) follows directly from (29) and (31). The lemma is proved.

For $f \in L_{\alpha,\beta}$ we set $S_m(f) = \sum_{k=0}^{m} c_k(f)\, J_k$ $(m+1 \in N)$.

LEMMA 8. *Let $r \in N$, $f \in \Omega_r$. The equality*

$$f - S_{r-1}(f) = \mathscr{D}_r(f) * \Phi_r \tag{32}$$

holds almost everywhere on $(-1, 1)$ *with respect to Lebesgue measure* (*in the sequel we shall use he abbreviation "a.e. on* $(-1, 1)$").

This lemma is an immediate consequence of (4), (15), and (27).

Remark 1. As we have seen, for the proof of (32), and consequently for all the subsequent considerations, it was crucial that, according to (27), the Fourier–Jacobi coefficients of $\mathscr{D}_r(f)$ turned out to be certain multiples of the Fourier–Jacobi coefficients of f. It can be proved that in order for a linear differential operator of order $2r$ to possess this property it is necessary and sufficient that the Jacobi polynomials $\{J_n^{(\alpha, \beta)}\}_0^\infty$ be eigenfunctions of the operator. On the other hand, it can be proved that such a differential operator can be represented as a linear combination with constant coefficients of the operators $\mathscr{D}_k$. Therefore, we have some reason to consider the operators $\mathscr{D}_k$ to be the simplest among all linear differential operators possessing the above mentioned property and, consequently, our choice of the operators $\mathscr{D}_k$ is to some extent justified.

6. THE MAIN THEOREM

Let $r \in N$, $p \in [1, \infty]$. We denote

$$\Omega_{r, p} = \{f \colon f \in \Omega_r, \mathscr{D}_r(f) \in L_{p; \alpha, \beta}\},$$

$$\tilde{\Omega}_{r, p} = \{f \colon f \in \Omega_{r, p}, E_{r-1}(\mathscr{D}_r(f))_{p; \alpha, \beta} \leqslant 1\}.$$

It is clear that $\Omega_{r, 1} = \Omega_r$. For $f \in \tilde{\Omega}_{r, p}$ we introduce

$$V_{f, r, p} = \{Q_{r-1, p}(f) \colon Q_{r-1, p}(f) \in H_{r-1}, \|\mathscr{D}_r(f) + Q_{r-1, p}(f)\|_{p; \alpha, \beta} \leqslant 1\};$$

obviously, $V_{f, r, p} \neq \varnothing$. For $f \in L_{\alpha, \beta}$, $n + 1 \in N$ we denote $R_n(f) = f - S_n(f)$.

LEMMA 9. *Let $r \in N$, $p \in [1, \infty]$. The set*

$$\{\mathscr{D}_r(f) + Q_{r-1, p}(f) : f \in \tilde{\Omega}_{r, p}, Q_{r-1, p}(f) \in V_{f, r, p}\}$$

is the unit ball in $L_{p; \alpha, \beta}$ with center at O.

Proof. First we note that, due to the definition of $V_{f,r,p}$, we have $\|\mathcal{D}_r(f)+Q_{r-1,p}(f)\|_{p;\alpha,\beta}\leqslant 1$ $\forall f\in\tilde{\Omega}_{r,p}$, $\forall Q_{r-1,p}\in V_{f,r,p}$. Let $\varphi\in L_{p;\alpha,\beta}$, $\|\varphi\|_{p;\alpha,\beta}\leqslant 1$. We will show that there exist $f\in\tilde{\Omega}_{r,p}$ and $Q_{r-1,p}(f)\in V_{f,r,p}$ such that $\mathcal{D}_r(f)+Q_{r-1,p}(f)=\varphi$. We set

$$f(x)=\Gamma^{-2}(r)\int_0^x (x-u)^{r-1}\,(p(u))^{-1}\,(1-u^2)^{-r}\,du$$

$$\times\int_{-1}^u (u-t)^{r-1}\,p(t)\,R_{r-1}(\varphi;t)\,dt. \tag{33}$$

We will prove that $f\in\Omega_r$. It follows from (33) that $\forall x\in(-1,1)$ we have

$$f^{(r)}(x)=\Gamma^{-1}(r)(p(x))^{-1}\,(1-x^2)^{-r}\int_{-1}^x (x-t)^{r-1}\,p(t)\,R_{r-1}(\varphi;t)\,dt$$

or

$$p(x)(1-x^2)^r\,f^{(r)}(x)=\Gamma^{-1}(r)\int_{-1}^x (x-t)^{r-1}\,p(t)\,R_{r-1}(\varphi;t)\,dt. \tag{34}$$

From (34) we obtain by mathematical induction that $\forall x\in(-1,1)$ and $k=\overline{0,r-1}$ we have

$$(p(x)(1-x^2)^r\,f^{(r)}(x))^{(k)}=\Gamma^{-1}(r-k)\int_{-1}^x (x-t)^{r-1-k}\,p(t)\,R_{r-1}(\varphi;t)\,dt. \tag{35}$$

Since $R_{r-1}(\varphi)\perp H_{r-1}$, it follows from (35) that for $k=\overline{0,r-1}$ we have

$$\psi_{f,r}^{[k]}(x)=\begin{cases}(p(x)(1-x^2)^r\,f^{(r)}(x))^{(k)}, & |x|<1,\\ 0, & |x|=1,\end{cases}\quad\in AC[-1,1].$$

We have proved that $f\in\Omega_r$. It follows from (35) that if we take $k=r-1$ then $\forall x\in(-1,1)$ the following equality holds:

$$(p(x)(1-x^2)^r\,f^{(r)}(x))^{(r-1)}=\int_{-1}^x p(t)\,R_{r-1}(\varphi;t)\,dt.$$

This in turn implies that a.e. on $(-1,1)$ we have

$$(p(x)(1-x^2)^r\,f^{(r)}(x))^{(r)}=p(x)\,R_{r-1}(\varphi;x),$$

i.e.

$$\mathcal{D}_r(f)=\varphi-S_{r-1}(\varphi)$$

or

$$\varphi = \mathscr{D}_r(f) + S_{r-1}(\varphi). \tag{36}$$

Since $\|\varphi\|_{p;\,\alpha,\,\beta} \leqslant 1$, it follows from (36) that $E_{r-1}(\mathscr{D}_r(f))_{p;\,\alpha,\,\beta} \leqslant 1$, so that $f \in \tilde{\Omega}_{r,\,p}$; it remains to set $Q_{r-1,\,p}(f) = S_{r-1}(\varphi)$. The lemma is proved.

THEOREM 1. *If* r, $n+1 \in N$, $n \geqslant r-1$, *and* r, α, q *are consistent, then*

$$\sup\{E_n(f)_{q;\,\alpha,\,\beta} \cdot (E_n(\mathscr{D}_r(f)))^{-1}_{1;\,\alpha,\,\beta} \colon f \in \Omega_r\} = E_n(\Phi_r)_{q;\,\alpha,\,\beta} \tag{37}$$

(*here and below, the symbol* $\frac{0}{0}$ *is considered to have the value* 0).

Proof. First we notice that, since r, α, q are consistent, it follows from (32), Lemma 4, and property (2) of the g.c. that $\Omega_r \subset L_{q;\,\alpha,\,\beta}$. Taking into account (32), Lemma 9, and (13), we obtain

$$\begin{aligned}
&\sup\{E_n(f)_{q;\,\alpha,\,\beta} \colon f \in \tilde{\Omega}_r\}\\
&\quad = \sup\{E_n((\mathscr{D}_r(f) * \Phi_r) + S_{r-1}(f))_{q;\,\alpha,\,\beta} : f \in \tilde{\Omega}_r\}\\
&\quad = \sup\{E_n((\mathscr{D}_r(f) * \Phi_r))_{q;\,\alpha,\,\beta} \colon f \in \tilde{\Omega}_r\}\\
&\quad = \sup\{E_n((\mathscr{D}_r(f) + Q_{r-1,\,1}(f)) * \Phi_r)_{q;\,\alpha,\,\beta} : f \in \tilde{\Omega}_r,\ Q_{r-1,\,1} \in V_{f,\,r,\,1}\}\\
&\quad = \sup\{E_n(\varphi * \Phi_r)_{q;\,\alpha,\,\beta} \colon \|\varphi\|_{1;\,\alpha,\,\beta} \leqslant 1\} = E_n(\Phi_r)_{q;\,\alpha,\,\beta}.
\end{aligned} \tag{38}$$

It follows from (38) that

$$\begin{aligned}
&\sup\{E_n(f)_{q;\,\alpha,\,\beta} \cdot (E_n(\mathscr{D}_r(f))_{1;\,\alpha,\,\beta})^{-1} : f \in \tilde{\Omega}_r\}\\
&\quad \geqslant \sup\{E_n(f)_{q;\,\alpha,\,\beta} \cdot (E_{r-1}(\mathscr{D}_r(f))_{1;\,\alpha,\,\beta})^{-1} : f \in \tilde{\Omega}_r\}\\
&\quad \geqslant \sup\{E_n(f)_{q;\,\alpha,\,\beta} \colon f \in \tilde{\Omega}_r\} = E_n(\Phi_r)_{q;\,\alpha,\,\beta}.
\end{aligned} \tag{39}$$

On the other hand, making use of (32) and Lemma 3, we obtain that $\forall f \in \Omega_r$ we have

$$E_n(f)_{q;\,\alpha,\,\beta} \leqslant E_n(\mathscr{D}_r(f))_{1;\,\alpha,\,\beta} \cdot E_n(\Phi_r)_{q;\,\alpha,\,\beta}, \tag{40}$$

which implies

$$\sup\{E_n(f)_{q;\,\alpha,\,\beta} \cdot (E_n(\mathscr{D}_r(f))_{1;\,\alpha,\,\beta})^{-1} : f \in \tilde{\Omega}_r\} \leqslant E_n(\Phi_r)_{q;\,\alpha,\,\beta}. \tag{41}$$

Comparing (39) and (41), we conclude that

$$\sup\{E_n(f)_{q;\,\alpha,\,\beta} \cdot (E_n(\mathscr{D}_r(f))_{1;\,\alpha,\,\beta})^{-1} : f \in \tilde{\Omega}_r\} = E_n(\Phi_r)_{q;\,\alpha,\,\beta}. \tag{42}$$

Since the ratio under the sup symbol is homogeneous with respect to f, the equality (42) implies (37). The theorem is proved.

7. SOME ESTIMATES FOR $E_n(\Phi_r)_{q;\alpha,\beta}$

In connection with Theorem 1 it is timely to discuss the question of the estimation of the quantities $E_n(\Phi_r)_{q;\alpha,\beta}$. In order to prove the following statement, we need

LEMMA 10. *If* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $1 \leqslant p < q \leqslant \infty$, $f \in L_{p;\alpha,\beta}$ *and*

$$\sum_{n=1}^{\infty} n^{2(1/p-1/q)(\alpha+1)-1} E_n(f)_{p;\alpha,\beta} < \infty,$$

then $f \in L_{q;\alpha,\beta}$ *and for* $n \in N$ *we have*

$$E_n(f)_{q;\alpha,\beta} \leqslant C_3(p,q,\alpha,\beta) \left\{ n^{2(1/p-1/q)(\alpha+1)} E_n(f)_{p;\alpha,\beta} + \sum_{\nu=n+1}^{\infty} \nu^{2(1/p-1/q)(\alpha+1)-1} E_\nu(f)_{p;\alpha,\beta} \right\}. \tag{43}$$

A similar statement for the best approximation of 2π-periodic functions by trigonometric polynomials was proved in [15]. For the approximation of functions, defined on $[-1, 1]$, by algebraic polynomials, see [13, 24]. In our case the line of reasoning is the same as in [13, 15, 24]; for the proof of Lemma 10 the only specific piece of information we need is that for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $1 \leqslant p < q \leqslant \infty$, $Q_n \in H_n$ $(n+1 \in N)$ one has

$$\|Q_n\|_{q;\alpha,\beta} \leqslant C_4(p,q,\alpha,\beta)\, n^{2(1/p-1/q)(\alpha+1)}\, \|Q_n\|_{p;\alpha,\beta};$$

(see [5]). We leave the details of the proof to the reader.

STATEMENT 1. *If* $2 \leqslant q \leqslant \infty$, $n \in N$, $n \geqslant r-1$, $r > (1+\alpha)(1-\frac{1}{q})$, *then we have the estimate*

$$E_n(\Phi_r)_{q;\alpha,\beta} \leqslant C_5(r,q,\alpha,\beta) \cdot n^{2(1+\alpha)(1-q^{-1})-2r}; \tag{44}$$

moreover,

$$E_n(\Phi_r)_{2;\alpha,\beta} \sim n^{1+\alpha-2r} \qquad \left(r > \frac{1+\alpha}{2},\, n \to \infty \right) \tag{45}$$

(*for two sequences* $\{a_n\}_{n=m}^{\infty}$, $\{b_n\}_{n=m}^{\infty}$ *of positive numbers we write* $a_n \sim b_n$ *as* $n \to \infty$ *if there exist two positive constants* C_6 *and* C_7 *such that for* $n \geqslant m$ *we have* $C_6 \leqslant \frac{a_n}{b_n} \leqslant C_7$).

Proof. To prove (45) we take into account that

$$E_n(\Phi_r)_{2,\alpha,\beta} = \left\{ \sum_{k=n+1}^{\infty} C_1^2(k,r,\alpha,\beta) \cdot \|J_k\|_C^2 \right\}^{1/2}$$

and that

$$C_1(k,r,\alpha,\beta) \sim k^{-2r} (k \to \infty), \qquad \|J_k\|_C \sim k^{\alpha+1/2} \qquad (k \to \infty).$$

In order to prove (44) one can apply Lemma 10 when $p = 2$, $2 < q \leqslant \infty$ and make use of (45). The statement is proved.

8. THE CASE $\alpha = \beta = 0$

In this case we will give additional information about the constant that appears in the extremal relation (37). We will use the notations

$$\Phi_{r;0,0} = \Phi_{r;0}, \qquad \Omega_{r;0,0} = \Omega_{r;0},$$

$$\mathscr{D}_{r;0,0} = \mathscr{D}_{r;0}, \qquad E_n(f)_{p;0,0} = E_n(f)_{p;0}.$$

We set also

$$R_{n;0}(f) = \sum_{k=0}^{n} c_k^{0,0}(f)\, J_k^{(0,0)}, \qquad n+1 \in N.$$

LEMMA 11. *Let $r \in N$. The following equality holds on $[-1, 1)$:*

$$\Phi_{r;0}(t) = 2^{-r}\Gamma^{-2}(r)(1-t)^{r-1}\ln(1-t) + A_{r-1}(t), \qquad A_{r-1} \in H_{r-1}. \tag{46}$$

We omit the simple proof of this lemma.

Let T_n $(n \in N)$ be the set of all ordered n-tuples with real coordinates.

LEMMA 12. *Let $r \in N$. We set*

$$\Psi_r(d,t) = \Phi_{r;0}(t) - \sum_{k=0}^{n} d_k t^k,$$

where $d = (d_0, ..., d_n) \in T_{n+1}$, $n+1 \in N$. If $n \geqslant r-1$, then $\forall d \in T_{n+1}$ the function $\Psi_r(d,\cdot)$ has at most $n+1$ zeros in $(-1, 1)$, multiplicities included.

Proof. The proof is by contradiction, i.e. we assume that $\Psi_r(d, \cdot)$ has at least $n+2$ zeros in $(-1, 1)$, multiplicities included. Then, by Rolle's theorem, $\Psi_r^{(n+1)}(d, \cdot)$ has at least one zero in $(-1, 1)$. On the other hand, from (46) we have

$$\Psi_r^{(n+1)}(d, t) = \Phi_{r;0}^{(n+1)}(t) = 2^{-r}\Gamma^{-1}(r) \cdot (n+1-r)!\,(1-t)^{-n-2+r}.$$

The obtained contradiction proves the lemma.

Lemma 13. *If $n+1,\ r \in N,\ n \geqslant r-1$, then we have*

$$E_n(\Phi_{r;0})_{1;0} \stackrel{\text{def}}{=} M_{r,n} = 4(n+2)\,\Gamma\left(r+\frac{1}{2}\right)\Gamma^{-1}(r)\,\pi^{-1/2}\sum_{v=0}^{\infty}\frac{((2v+1)(n+2)-1-r)!}{((2v+1)(n+2)+r)!}. \tag{47}$$

Proof. We set $t_k = \cos\frac{k\pi}{n+2}$ $(k=\overline{1, n+1})$. Let $P_n^* \in H_n$ interpolate $\Phi_{r;0}$ with the nodes t_k $(k=\overline{1, n+1})$. We introduce $\Phi_r^* = \Phi_{r;0} - P_n^*$. It is clear that $\Phi_r^*(t_k) = 0$ $(k=\overline{1, n+1})$; therefore, due to Lemma 12, the function Φ_r^* has no zeroes in $(-1, 1)$ other than t_k $(k=\overline{1, n+1})$. Moreover, all these zeros are simple. We conclude that the function $\Phi_r^*(t)\,\phi_n(t)$, where $\phi_n(t) = \operatorname{sign}\sin((n+2)\arccos t)$, retains its sign on $(-1, 1)$. Making use of theorem of A. A. Markov [2, p. 84], we obtain that P_n^* is the polynomial from H_n of best approximation to $\Phi_{r;0}$ in the L-metric; moreover

$$M_{r,n} = \left|\int_{-1}^{1} \Phi_{r;0}(t)\,\phi_n(t)\,dt\right|. \tag{48}$$

Taking into account (48), (46) and the well-known relation

$$\int_{-1}^{1} Q_n(t)\,\phi_n(t)\,dt = 0 \qquad \forall Q_n \in H_n, \tag{49}$$

we deduce that

$$M_{r,n} = 2^{-r}\Gamma^{-2}(r)\left|\int_{-1}^{1}(1-t)^{r-1}\ln(1-t)\,\phi_n(t)\,dt\right|. \tag{50}$$

Applying the equality

$$\operatorname{sign}\sin t = 4\pi^{-1}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\,t}{2k+1},$$

we derive that

$$\int_{-1}^{1} (1-t)^{r-1} \ln(1-t)\, \phi_n(t)\, dt$$

$$= 4\pi^{-1} \sum_{v=0}^{\infty} (2v+1)^{-1} \int_{-1}^{1} (1-t)^{r-1} \ln(1-t)\, \phi_{v,n}(t)\, dt, \tag{51}$$

where $\phi_{v,n}(t) = \sin((2v+1)(n+2) \arccos t)$. In view of the formula

$$(1-t^2)^{-1/2}\, \phi_{v,n}(t) = \pi^{1/2} 2^{-1/2} J^{(1/2,\,1/2)}_{(2v+1)(n+2)-1}(t),$$

we obtain

$$I_{v,n} \stackrel{\text{def}}{=} \int_{-1}^{1} (1-t)^{r-1} \ln(1-t)\, \phi_{v,n}(t)\, dt$$

$$= \pi^{1/2} \cdot 2^{-1/2} \int_{-1}^{1} (1-t^2)^{1/2}\, (1-t)^{r-1} \ln(1-t)\, J^{(1/2,\,1/2)}_{(2v+1)(n+2)-1}(t)\, dt. \tag{52}$$

Let $m = (2v+1)(n+2)-1$. Making use of Rodrigues' formula for $J_m^{(1/2,\,1/2)}(t)$ and integrating m times by parts, after simple transformations we obtain

$$I_{v,n} = (-1)^r\, \Gamma(r)\, 2^r \sqrt{\pi}\, \Gamma(r+2^{-1})(m-r)!\, (m+1)((m+r+1)!)^{-1}. \tag{53}$$

Equality (47) follows directly from (50)–(53). The proof is complete.

Remark 2. Equality (49) is closely related to the fact that the Chebyshev polynomial of the second kind with leading coefficient 1, namely

$$U_n(x) = 2^{-n}(1-x^2)^{-1/2} \sin((n+1) \arccos x)$$

minimizes the integral $\int_{-1}^{1} |Q_n(x)|\, dx$ among all polynomials $Q_n \in H_n$ with leading coefficient 1.

Remark 3. In fact, in Lemma 13 we have proved that for $n+1,\ r \in N$, $n \geqslant r-1$ the following equality holds:

$$E_n((1-x)^{r-1} \ln(1-x))_{1;\,0}$$

$$= 2^{r+2}(n+2)\, \Gamma(r)\, \Gamma\left(r+\frac{1}{2}\right) \pi^{-1/2} \sum_{v=0}^{\infty} \frac{((2v+1)(n+2)-1-r)!}{((2v+1)(n+2)+r)!}.$$

In particular, if $n+1 \in N$, then

$$E_n(\ln(1-x))_{1;0} = 4 \sum_{v=0}^{\infty} ((2v+1)((2v+1)^2 (n+2)^2 - 1))^{-1};$$

for comparison, we quote the following equality [33, p. 462]:

$$E_n(\ln(1-x))_{1;0} = (1+o(1)) \cdot 4n^{-2} \sum_{v=0}^{\infty} (2v+1)^{-3}, \qquad n \in N.$$

COROLLARY 2. *The following equality holds asymptotically as* $n \to \infty$:

$$M_{r,n} = 4\Gamma(r+\tfrac{1}{2})\, \pi^{-1/2} \Gamma^{-1}(r)(n+2)^{-2r} \sum_{v=0}^{\infty} (2v+1)^{-2r-1} + C_{10}(r,n)(n+2)^{-2r-2}; \tag{54}$$

here,

$$C_{10}(r,n) < 4\Gamma\left(r+\frac{1}{2}\right) \Gamma^{-1}(r)\, \pi^{-1/2} r^3 \times \left(\frac{(r+1)^2}{2r+1} + \frac{1}{4} \sum_{v=1}^{\infty} \frac{1}{(v+1)^r\, v^r (2v+1)^3}\right). \tag{55}$$

This statement can be derived easily from (47).

COROLLARY 3. *The following estimate holds: if* $n \in N$, $1 \leqslant q \leqslant 2$, $r > 1 - \frac{1}{q}$, *then*

$$E_n(\Phi_{r;0})_{q;0} \leqslant C_{11}(r,q)\, n^{2(1-q^{-1}-r)}. \tag{56}$$

In order to prove this statement we apply Lemma 10 *for* $\alpha = \beta = 0$, $p = 1$, $1 < q \leqslant 2$ *and take into account* (54) *and* (55). *Thus, in the case* $\alpha = \beta = 0$ *the estimate* (44) *holds if* $1 \leqslant q \leqslant \infty$, $r > 1 - \frac{1}{q}$.

REMARK 4. *The following relation holds: if* $r \in N$, $r \geqslant 2$, *then*

$$E_n(\Phi_{r;0})_C \sim n^{-2r+2}.$$

Proof. Due to (44), it is sufficient to prove that

$$E_n(\Phi_{r;0})_C \geqslant C_{12}(r)\, n^{-2r+2}. \tag{57}$$

In order to prove (57) one can use the following inequality

$$E_n(f)_C \geqslant C_{13} \cdot n^{1/2} \sum_{k=n+1}^{2n+2} c_k^{0,0}(f), \tag{58}$$

where $C_{13} > 0$ is an absolute constant. Inequality (58) holds if $f \in C$, $n \in N$, $c_k^{0,0}(f) \geqslant 0$ $(k+1 \in N)$, [25].

Corollary 4. *If* r, $n+1 \in N$, $n \geqslant r-1$, *then the following sharp inequality holds on* $\Omega_{r,0}$:

$$E_n(f)_{1;0} \leqslant M_{r,n} \cdot E_n(\mathscr{D}_r(f))_{1;0}. \tag{59}$$

Here and in the sequel by the sharpness of an inequality we mean that the constant factor in the right-hand side cannot be replaced by a smaller one on the whole class of functions. This corollary follows from Theorem 1 and (47).

Lemma 14. *If* $n+1$, $r \in N$, $n \geqslant r-1$, *then the following equality holds*:

$$E_n(\Phi_{r;0})_{2;0} = (2(2r-1))^{-1/2} \cdot \frac{\Gamma(n-r+2)}{\Gamma(n+r+1)}. \tag{60}$$

Proof. It follows from Lemma 5 that

$$\begin{aligned} E_n(\Phi_{r;0})_{2;0} &= \left\{ \sum_{k=n+1}^{\infty} \frac{\Gamma^2(k-r+1)(2k+1)}{2\Gamma^2(k+r+1)} \right\}^{1/2} \\ &= \frac{1}{\sqrt{2}} \left\{ \sum_{k=n+1}^{\infty} \frac{((k-r)!)^2\,((k+r)+(k-r+1))}{((k+r)!)^2} \right\}^{1/2} \\ &= \frac{1}{\sqrt{2}} \left\{ \sum_{k=n+1}^{\infty} \frac{((k-r)!)^2}{(k+r)((k+r-1)!)^2} \right. \\ &\quad \left. + \sum_{k=n+1}^{\infty} \frac{((k-r+1)!)^2}{(k-r+1)((k+r)!)^2} \right\}^{1/2} \\ &= \frac{1}{\sqrt{2}} \left\{ \sum_{k=n+1}^{\infty} (k+r)^{-1} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \right. \\ &\quad \left. + \sum_{k=n+1}^{\infty} (k-r+1)^{-1} \prod_{i=-r+2}^{r} (k+i)^{-2} \right\}^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} \left\{ \sum_{k=n+1}^{\infty} \frac{(k+r)-(k-r+1)}{(k+r)(2r-1)} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \right. \\
&\quad \left. + \sum_{k=n+1}^{\infty} \frac{(k+r)-(k-r+1)}{(k-r+1)(2r-1)} \prod_{i=-r+2}^{r} (k+i)^{-2} \right\}^{1/2} \\
&= \frac{1}{\sqrt{2}\sqrt{2r-1}} \left\{ \sum_{k=n+1}^{\infty} \frac{k+r}{k-r+1} \prod_{i=-r+2}^{r} (k+i)^{-2} \right. \\
&\quad - \sum_{k=n+1}^{\infty} \prod_{i=-r+2}^{r} (k+i)^{-2} + \sum_{k=n+1}^{\infty} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \\
&\quad \left. - \sum_{k=n+1}^{\infty} \frac{k-r+1}{k+r} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \right\}^{1/2} \\
&= \frac{1}{\sqrt{2}\sqrt{2r-1}} \left\{ \sum_{j=n+2}^{\infty} \frac{j+r-1}{j-r} \prod_{i=-r+2}^{r} (j+i-1)^{-2} \right. \\
&\quad - \sum_{j=n+2}^{\infty} \prod_{i=-r+2}^{r} (j+i-1)^{-2} + \sum_{j=n+1}^{\infty} \prod_{i=-r+1}^{r-1} (j+i)^{-2} \\
&\quad \left. - \sum_{k=n+1}^{\infty} \frac{k-r+1}{k+r} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \right\}^{1/2} \\
&= \frac{1}{\sqrt{2}\sqrt{2r-1}} \left\{ \prod_{i=-r+1}^{r-1} (n+1+i)^{-2} \right. \\
&\quad + \sum_{j=n+2}^{\infty} \frac{j+r-1}{j-r} \prod_{i=-r+2}^{r} (j+i-1)^{-2} \\
&\quad \left. - \sum_{k=n+1}^{\infty} \frac{k-r+1}{k+r} \prod_{i=-r+1}^{r-1} (k+i)^{-2} \right\}^{1/2}. \qquad (61)
\end{aligned}
$$

Now we will prove that

$$
\begin{aligned}
B(n,r) &\overset{\text{def}}{=} \sum_{j=n+2}^{\infty} \frac{j+r-1}{j-r} \prod_{i=-r+2}^{r} (j+i-1)^{-2} \\
&\quad - \sum_{k=n+1}^{\infty} \frac{k+r-1}{k+r} \prod_{i=-r+1}^{r-1} (k+i)^{-2} = 0. \qquad (62)
\end{aligned}
$$

In fact,

$$B(n, r) = \sum_{k=n+1}^{\infty} \frac{k+r}{k+1-r} \prod_{i=-r+2}^{r} (k+i)^{-2}$$

$$- \sum_{k=n+1}^{\infty} \frac{k-r+1}{k+r} \prod_{i=-r+1}^{r-1} (k+i)^{-2}$$

$$= \sum_{k=n+1}^{\infty} \left(\frac{1}{(k-r+1) \prod_{i=-r+2}^{r-1} (k+i)^2 (k+r)} - \frac{1}{(k+r)(k-r+1) \prod_{i=-r+2}^{r-1} (k+i)^2} \right) = 0,$$

so that equality (62) is proved. Equality (60) follows at once from (61) and (62). The lemma is proved.

Corollary 5. *For $n+1$, $r \in N$, $n \geqslant r-1$ the following sharp inequality holds on $\Omega_{r;0}$:*

$$E_n(f)_{2;0} \leqslant (2(2r-1))^{-1/2}\, \Gamma(n-r+2)\, \Gamma^{-1}(n+r+1)\, E_n(\mathcal{D}_r(f))_{1;0}. \tag{63}$$

This statement follows directly from Theorem 1 and Lemma 14.

We formulate the next statement without proof.

Remark 5. If $n+1$, $r \in N$, $r > 1$, $n \geqslant r-1$, then the following sharp inequality holds on $\Omega_{r,0}$:

$$\|R_{n;0}(f)\|_C \leqslant \frac{n+1}{2(r-1)} \left(\prod_{i=-r+2}^{r} (n+i) \right)^{-1} \cdot E_n(\mathcal{D}_{r;0}(f))_{1;0}.$$

Remark 6. The inequality (40) is in fact a version of the Jackson-type second theorem where in the case of the approximation of 2π-periodic functions by trigonometric polynomials the best approximation of a function is estimated from above by the best approximation of a derivative of the function.

We will consider some particular cases of Corollary 4.

(1) If $n+1 \in N$, then $\forall f \in \Omega_{1;0}$ we have

$$E_n(f)_{1;0} \leqslant M_{1;n} \cdot E_n(\mathcal{D}_1(f))_{1;0}.$$

From the formula (47) it is easy to derive the estimate

$$M_{1,n} < \frac{2.11}{(n+2)(n+1)} \qquad (n+1 \in N),$$

so that $\forall n+1 \in N$ and $\forall f \in \Omega_{1;0}$ the following Jackson-type second theorem is valid:

$$E_n(f)_{1;0} \leqslant \frac{2.11}{(n+2)(n+1)} \cdot E_n(\mathscr{D}_1(f))_{1;0}$$
$$\leqslant \frac{2.11}{(n+2)(n+1)} [E_n((1-x^2) f''(x))_{1;0} + 2E_n(xf'(x))_{1;0}].$$

(2) If $n \in N$, then $\forall f \in \Omega_{2;0}$ we have

$$E_n(f)_{1;0} \leqslant M_{2;n} \cdot E_n(\mathscr{D}_2(f))_{1;0}.$$

It is easy to derive from (54) and (55) that $\forall n \in N$ we have

$$M_{2,n} < 2.01(n+2)^{-4} + 29.31(n+2)^{-6},$$

so that $\forall n \in N$ and $\forall f \in \Omega_{2;0}$ the following version of the Jackson-type second theorem is valid

$$E_n(f)_{1;0} \leqslant \{2.01(n+2)^{-4} + 29.31(n+2)^{-6}\}\{E_n((1-x^2)^2 f^{(4)}(x))_{1;0}$$
$$+ 8E_n(x(1-x^2) f'''(x))_{1;0} + E_n((-4+12x^2) f''(x))_{1;0}\}.$$

9. ON SOME CLASSES OF FUNCTIONS

For $r \in N$ we introduce

$$W^r_L = \{f\colon f^{(r-1)} \in AC[-1, 1], \|f^{(r)}\|_{1;0} \leqslant 1\},$$
$$E_n(W^r_L)_L = \sup \{E_n(f)_{1;0} \colon f \in W^r_L\}.$$

V. A. Kofanov [14] has found the exact value of $E_n(W^r_L)_L$. It seems relevant to compare the classes W^{2r}_L and $\Omega_{r;0}$. Obviously, $W^{2r}_L \subset \Omega_{r;0}$. We will show now that $W^{2r}_L \neq \Omega_{r;0}$. In fact, it is easy to verify that the function $f_\mu(x) = (1-x^2)^\mu \in \Omega_{r;0} \backslash W^{2r}_L$ if $r-1 < \mu < 2r-1$. As a matter of fact, the class $\Omega_{r;0}$ contains functions that have stronger singularities at the points $x = \pm 1$ than the functions from W^{2r}_L. Thus, the functional class on which the inequality (59) holds is broader than the class W^{2r}_L.

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